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A modular damage model for quasi-brittle solids – interaction between initial and induced anisotropy

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Summary This paper proposes a three-dimensional thermodynamically controlled damage model for a wide class of quasi-brittle materials, the modelling strategy being a continuation of the earlier work, [12–15]. The purpose is to keep an existing modular structure and to introduce new features to its framework. These are: (i) a thrifty insertion of initial orthotropy, (ii) the absence of irreversible strain after loading/unloading cycles (in opposition to rock-like materials described by the initial model) and especially (iii) the competition between initial orthotropy and anisotropy induced by mesocrack growth. The proposed innovation consists in adding second-order fabric tensors in conjunction with a damage tensor in the expression of the thermodynamic potential. Experimental data for a test composite material are simulated by this approach.

Keywords Quasi-brittle solid, Modular model, Damage, Initial orthotropy, Fabric tensor, Induced anisotropy

1 Introduction

In the field of brittle materials (e.g. rock-like solids, concrete, ceramic matrix composites), a number of nonlinear models involving explicitly damage effects by mesocrack growth appeared in the literature during the last two decades. Among them, two main classes can be roughly distinguished: micromechanical approach (see, e.g. [1–3] for rocks and concrete or [4, 5] for composites) and phenomenological one (e.g. for composites, [6–8]). The first class of models, though physically well-grounded, suffers, in general, from inherent complexity and restricted applicability (frequently to 2D problems), which is sometimes related to problems of numerical integration. One can consult [9] for a general discussion on models for brittle composites. In this context, a modelling strategy has been advanced, attempting to promote a sort of compromise by linking several methodological tools: tensor functions representation, thermodynamics, internal variables. It remains macromechanically based while, at the same time, most of the model ingredients are strongly enriched with some micromechanical acquisitions. This way has been followed recently by some authors, see e.g. [10, 11]. One of the advantages of this macro/micromechanical interpretation strategy consists in putting forward modular and 'open' structures of the resulting models. For example, the model of anisotropic damage for initially isotropic brittle solids involves a so called 'basic version', followed by more advanced multi-

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dissipative models of growing complexity and accounting for further nonlinear features. The main steps of the development of this model are given as follows:

- Within a thermodynamic framework of internal variables, the basic level, [12], describes the gradual degradation by mesocrack growth, inducing a number of effects such as secondary anisotropy and dilatancy.
- Under compressive loading, favorably oriented cracks may close, leading to a stiffness–recovery phenomenon. This 'unilateral' effect may be exhibited at cyclic tension-and–compression tests. The basic version takes into account that created cracks definitely influence the effective properties. The second modular unit, [13], proposes a rigorous formulation of the stiffness recovery due to crack closure, based on micromechanical studies and avoiding stress discontinuities during the crack opening/closure transition.
- The previous stage concerns the normal unilateral effect: the stiffness is recovered in the direction normal to the closed crack set, while the shear (tangential) modulus remains affected by damage (as if cracks were perfectly lubricated). The third step, [14], models the dissipative frictional locking or sliding on rough crack lips (when closed), inducing the recovery of shear moduli and complex hysteretic effects during cyclic torsion, for instance. This third modular unit takes into account the coupling effects between damage and sliding in a 3D framework.

The whole model includes the three stages, covers thus a large number of events encountered in quasi-brittle materials. Still, its macroscopic formulation remains simple and adapted to structural analysis. An emphasis has been put on the identification of a small number (nine) of parameters and on the convenience of the integration scheme. Many examples can be found in [12–15].

The purpose of the present paper is to further extend this modular-damage approach by reconsidering two mechanical features concerning its basic segment. On the one hand, while accounting for the anisotropy induced by the mesocrack growth, the model, given in [12] in its first version, considers the material initially isotropic. This assumption is found too restrictive for a number of materials (e.g. sedimentary rocks, fiber-reinforced composites, etc.). Here, the introduction of initial orthotropy is considered by using a parsimonious optimised method consisting in combining three transverse isotropy operators (fabric tensors). One obtains but a small number of constants to be determined, compared to other approaches. On the other hand, the basic version of the model [12] recalled above puts forward the role played by damage-induced residual effects for rock-like solids as e.g. residual strain exhibited by tension – compression cycles. However, for a large family of brittle matrix composites (e.g. ceramic-matrix composites, [16]), this permanent stress/strain is not manifest, at least for matrix-cracking stage of degradation. Thus, this paper attempts to relax some residual damage-induced effects in order to cope with initially anisotropic materials without irreversible strain. It is worth noting that multiple matrix-cracking is the primary dissipative mechanism considered here. An important issue in this framework is the interaction of oriented damage (and respective secondary anisotropy) with an initial anisotropy. Some micromechanical studies, [17], address this problem and propose some tools to quantify the respective coupling. Those tools can be hardly exploited in a general 3D context. An alternative is thus advanced, involving conjunction of damage and fabric tensors to deal with coupling effects of primary anisotropy vs. those of evolving damage microcracking-induced one.

The paper is organised as follows: Sec. 2 treats the problem of initial anisotropy, while the aim of Sec. 3 is to model the mesocrack growth in the matrix inducing no significant irreversible stress/strain effects as well as its interaction with the initial anisotropy. As stressed above, this latter phenomenon is described by a formalism keeping the wish to propose an efficient alternative with respect to some micromechanical results. The capability of the model is validated by simulating tension tests on CMC plates.

2 Initial anisotropy

The basic version of the model mentioned in Sec. 1 concerns materials whose initial mechanical behaviour can be considered as isotropic. However, a number of materials do not fulfill this assumption and may exhibit a strong initial anisotropy due to the fabrication process, e.g. reinforced composites or sedimentary rocks. An alternative formulation preserving the methodological framework of the modelling allows one to take into account this primary anisotropy. In this section, this is orthotropy. The case of transverse isotropy is first studied, and then extended to orthotropy.

2.1
Case of transverse isotropy

The mechanical properties of a transversely isotropic material are identical in planes orthogonal to a given axial direction, so that the elastic stiffness matrix $\mathbf{C}^0$ takes the following expression if 1 is the symmetry axis (the classical Voigt convention is adopted):

$$\mathbf{C}^0 = \begin{bmatrix} C^0_{11} & C^0_{12} & C^0_{12} & 0 & 0 & 0 \\ & C^0_{22} & C^0_{23} & 0 & 0 & 0 \\ & & C^0_{22} & 0 & 0 & 0 \\ & & & \frac{1}{2}(C^0_{22} - C^0_{23}) & 0 & 0 \\ & \text{sym} & & & C^0_{55} & 0 \\ & & & & & C^0_{55} \end{bmatrix} . \tag{1}$$

The superscript "0" means initial (i.e. virgin material's) stiffness. Relevant literature deals with initial anisotropy by formulating a thermodynamic potential related to initial elasticity, w^0, within the framework of the tensor function representation theory, [18]: the form of w^0 must remain invariant with respect to the coordinate transformation expressing the material symmetries, and is built by the use of polynomial invariants (see [19] for the general formulation and a 2D application, or [20] for a 3D example). The present work is based on a method employing the same mathematical tools. Its particular feature is the *explicit* use of a second-order orientation tensor **A**, whose principal axes coincide with the material symmetry axes, unlike the above-cited works that implicitly formulate the thermodynamic potential in the orthotropy axes. The way chosen here to model the primary anisotropy of the material is to use "fabric tensors", which quantify directional data (see e.g. [21], for an exhaustive study on these tensors).

Let w^0 be the free energy of the undamaged material whose behaviour is restricted to small strain (thermal and rate-dependent effects are neglected here). Linear elasticity is assumed for this class of materials so that w^0 is a quadratic function of the strain tensor $\boldsymbol{\varepsilon}$. Classically, w^0 takes the following form in the case of initial isotropy:

$$w^0(\boldsymbol{\varepsilon}) = \frac{\lambda}{2}(\operatorname{tr}\boldsymbol{\varepsilon})^2 + \mu \operatorname{tr}(\boldsymbol{\varepsilon}.\boldsymbol{\varepsilon}) \ , \tag{2}$$

where λ and μ are the Lamé constants. For initially isotropic materials, the expression of w^0 thus contains only the strain tensor. The case of the transverse isotropy (and, more generally, anisotropy) requires a directional (fabric) tensor **A**. The problem is then equivalent to finding w^0 such that:

$$\begin{aligned} &w^0(\mathbf{Q}.\boldsymbol{\varepsilon}.\mathbf{Q}^t, \mathbf{Q}.\mathbf{A}.\mathbf{Q}^t) = w^0(\boldsymbol{\varepsilon}, \mathbf{A}), \quad \forall \mathbf{Q} \in O, \\ &w^0(\mathbf{Q}.\boldsymbol{\varepsilon}.\mathbf{Q}^t, \mathbf{A}) = w^0(\boldsymbol{\varepsilon}, \mathbf{A}), \quad \forall \mathbf{Q} \in T \ , \end{aligned} \tag{3}$$

where O is the full proper orthogonal group, i.e. $O = \{\mathbf{Q} | \mathbf{Q} \cdot \mathbf{Q}^t = \mathbf{Q}^t \cdot \mathbf{Q} = \mathbf{I}\}$, and $T \subset O$ is the symmetry group corresponding to transverse isotropy. Relations (3) mean that w^0 is an isotropic invariant of $\boldsymbol{\varepsilon}$ and **A**. The tensor function representation theory, [18], guides one to obtain the expression of w^0. This problem is similar to anisotropy resulting from changes in internal structure, [22].

As it can be found in the literature, [23, 24], let us define **A** by:

$$\mathbf{A} = \mathbf{a} \otimes \mathbf{a}, \quad ||\mathbf{a}|| = 1 \ , \tag{4}$$

where **a** is the transverse isotropy direction. The tensor **A** thus contains the information on anisotropy, e.g. the direction of the reinforcement in unidirectional fiber-reinforced composites. The dyadic product $\mathbf{a} \otimes \mathbf{a}$ is employed as w^0 must be an even function of **a**, since the sense of **a** is not significant.

According to [25], the expression of w^0 must include quadratic terms in **A**: transverse isotropic symmetry could not be represented by the solution of (3) if only linear terms in **A** entered w^0. After some calculation, the following expression of w^0 has been found to be sufficient to properly model the initial transverse isotropy:

$$w^0(\boldsymbol{\varepsilon}, \mathbf{A}) = \frac{a_1}{2}(\mathrm{tr}\,\boldsymbol{\varepsilon})^2 + a_2 \mathrm{tr}(\boldsymbol{\varepsilon}.\mathbf{A})\mathrm{tr}\,\boldsymbol{\varepsilon} + \frac{b_1}{2}[\mathrm{tr}(\boldsymbol{\varepsilon}\cdot\mathbf{A})]^2 + 2c_1 \mathrm{tr}(\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}) + 2c_2 \mathrm{tr}(\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}\cdot\mathbf{A}) \ . \tag{5}$$

It is worth noting that a rigorous use of the representation theory of tensor functions would lead to include in Eq. (5) terms of higher degree in **A**. However, the purpose here is to reach a compromise between the mathematical formulation and the number of parameters to be identified. The five parameters a_1, a_2, b_1, c_1, c_2 are easily identified from the coefficients of the stiffness tensor $\mathbf{C}^0$ (considered as experimentally known), by solving the following linear system:

$$\mathbf{C}^0 = \frac{\partial^2 w^0}{\partial\boldsymbol{\varepsilon}\partial\boldsymbol{\varepsilon}} \Rightarrow \begin{cases} C^0_{11} = a_1 + 4c_1 + 2(a_2 + 2c_2) + b_1, \\ C^0_{22} = a_1 + 4c_1, \\ C^0_{12} = a_1 + a_2, \\ C^0_{23} = a_1, \\ C^0_{55} = 2c_1 + c_2 \ . \end{cases} \tag{6}$$

2.2
Extension to orthotropy

The previous paragraph showed the transversely isotropic elasticity expressed with a single fabric tensor in the energy function w^0. This allows now to model the initial response of a given class of composites, for example those that are reinforced by unidirectional fibers. Most composites (e.g. woven composites) exhibit orthotropic behaviour. This paragraph aims at extending the case of transverse isotropy to orthotropy.

The modelling of the orthotropic behaviour is based here on the fact that orthotropy may be considered as equivalent to a combination of three transverse symmetries with respect to three orthogonal directions. Instead of a single fabric tensor, three directional operators $\mathbf{A}_i$ $(i = 1, 2, 3)$ enter the expression of w^0. Let $w^0(\boldsymbol{\varepsilon}, \mathbf{A}_i)$ be the free energy for the transversely isotropic material characterized by $\mathbf{A}_i$:

$$w^0(\boldsymbol{\varepsilon}, \mathbf{A}_i) = \frac{a^i_1}{2}(\mathrm{tr}\,\boldsymbol{\varepsilon})^2 + a^i_2 \,\mathrm{tr}(\boldsymbol{\varepsilon}.\mathbf{A}_i)\mathrm{tr}\,\boldsymbol{\varepsilon} + \frac{b^i_1}{2}[\mathrm{tr}(\boldsymbol{\varepsilon}.\mathbf{A}_i)]^2 + 2c^i_1 \,\mathrm{tr}(\boldsymbol{\varepsilon}.\boldsymbol{\varepsilon}) + 2c^i_2 \,\mathrm{tr}(\boldsymbol{\varepsilon}.\boldsymbol{\varepsilon}.\mathbf{A}_i) \ . \tag{7}$$

The following additive decomposition for orthotropy can be proposed:

$$\begin{aligned} w^0(\boldsymbol{\varepsilon}; \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3) = \sum_{i=1}^{3} w^0(\boldsymbol{\varepsilon}, \mathbf{A}_i) &= \left(\sum_{i=1}^{3} \frac{a^i_1}{2}\right)(\mathrm{tr}\,\boldsymbol{\varepsilon})^2 + \left(\sum_{i=1}^{3} 2c^i_1\right)\mathrm{tr}(\boldsymbol{\varepsilon}.\boldsymbol{\varepsilon}) \\ &+ \sum_{i=1}^{3}\left[a^i_2 \,\mathrm{tr}(\boldsymbol{\varepsilon}.\mathbf{A}_i)\mathrm{tr}\,\boldsymbol{\varepsilon} + \frac{b^i_1}{2}[\mathrm{tr}(\boldsymbol{\varepsilon}.\mathbf{A}_i)]^2 + 2c^i_2 \,\mathrm{tr}(\mathbf{A}_i.\boldsymbol{\varepsilon}.\boldsymbol{\varepsilon})\right] \ . \end{aligned} \tag{8}$$

The foregoing hypothesis that orthotropy can be expressed as a resultant action of three transverse isotropy operators is put forward by assuming that $\mathbf{A}_i$ are unit and mutually orthogonal tensors (i.e. $\sum_{i=1}^{3} \mathbf{A}_i = \mathbf{I}$). The two invariants $\mathrm{tr}(\boldsymbol{\varepsilon}.\boldsymbol{\varepsilon})$ and $(\mathrm{tr}\,\boldsymbol{\varepsilon})^2$ can thus be expressed as follows:

$$\begin{aligned} \mathrm{tr}(\boldsymbol{\varepsilon}.\boldsymbol{\varepsilon}) &= \sum_{i=1}^{3} \mathrm{tr}(\mathbf{A}_i.\boldsymbol{\varepsilon}.\boldsymbol{\varepsilon}), \\ (\mathrm{tr}\,\boldsymbol{\varepsilon})^2 &= \sum_{i=1}^{3} \mathrm{tr}(\mathbf{A}_i.\boldsymbol{\varepsilon})\mathrm{tr}\,\boldsymbol{\varepsilon} \ . \end{aligned} \tag{9}$$

Replacing $\mathrm{tr}(\boldsymbol{\varepsilon}.\boldsymbol{\varepsilon})$ and $(\mathrm{tr}\,\boldsymbol{\varepsilon})^2$ by Eq. (9) in Eq. (8), the free energy for initial orthotropic materials takes the following form:

$$w^0(\boldsymbol{\varepsilon}; \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3) = \sum_{i=1}^{3}\left[\bar{a}_i \mathrm{tr}\,\boldsymbol{\varepsilon}\,\mathrm{tr}(\mathbf{A}_i.\boldsymbol{\varepsilon}) + \bar{b}_i[\mathrm{tr}(\mathbf{A}_i.\boldsymbol{\varepsilon})]^2 + \bar{c}_i \,\mathrm{tr}(\mathbf{A}_i.\boldsymbol{\varepsilon}.\boldsymbol{\varepsilon})\right] \ . \tag{10}$$

It can be noted that since the three fabric tensors $\mathbf{A}_i$ are not independent in the case of orthotropy (they are orthogonal), the 15 material parameters entering Eq. (8) are related by following relationships:

$$\bar{a}_i = a_2^i + \sum_{k=1}^{3} \frac{a_1^k}{2},$$
$$\bar{b}_i = \frac{b_1^i}{2}, \tag{11}$$
$$\bar{c}_i = c_2^i + \sum_{k=1}^{3} 2c_1^k \; .$$

The 15 parameters thus reduce to nine constants, namely $\bar{a}_i, \bar{b}_i, \bar{c}_i$ ($i = 1, 2, 3$), whose values are determined in the same way as for the case of the transverse isotropy, i.e. by identifying the stiffness tensor components expressed in the orthotropy directions 1, 2, 3:

$$\mathbf{C}^0 = \frac{\partial^2 w^0}{\partial \varepsilon \partial \varepsilon} \Rightarrow \begin{cases} C_{11}^0 = 2\bar{a}_1 + 2\bar{b}_1 + 2\bar{c}_1, \\ C_{22}^0 = 2\bar{a}_2 + 2\bar{b}_2 + 2\bar{c}_2, \\ C_{33}^0 = 2\bar{a}_3 + 2\bar{b}_3 + 2\bar{c}_3, \\ C_{12}^0 = \bar{a}_1 + \bar{a}_2, \\ C_{13}^0 = \bar{a}_1 + \bar{a}_3, \\ C_{23}^0 = \bar{a}_2 + \bar{a}_3, \\ 2C_{44}^0 = \bar{c}_2 + \bar{c}_3, \\ 2C_{55}^0 = \bar{c}_1 + \bar{c}_3, \\ 2C_{66}^0 = \bar{c}_1 + \bar{c}_2 \; . \end{cases} \qquad C_{ij} = 0 \text{ otherwise.} \tag{12}$$

Remark 1
The solution proposed here (orthotropy modelled by three fabric tensors) is highly advantageous and differs from others found in the literature. For example, [25, 26] model orthotropy by a single tensor $\mathbf{A}$. According to the representation theory of tensor functions, the most general form of the stiffness tensor involving one fabric tensor $\mathbf{A}$ is:

$$\begin{aligned} C_{ijkl}^0 = {} & a_1\delta_{ij}\delta_{kl} + a_2(A_{ij}\delta_{kl} + A_{kl}\delta_{ij}) + a_3(\delta_{ij}A_{km}A_{ml} + A_{im}A_{mj}\delta_{kl}) + b_1A_{ij}A_{kl} \\ & + b_2(A_{ij}A_{km}A_{ml} + A_{in}A_{nj}A_{kl}) + b_3A_{im}A_{mj}A_{kn}A_{nl} + c_1(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \\ & + c_2(A_{ik}\delta_{jl} + A_{jk}\delta_{il} + A_{il}\delta_{jk} + A_{jl}\delta_{ik}) \\ & + c_3(A_{im}A_{mk}\delta_{jl} + A_{jm}A_{ml}\delta_{ik} + A_{im}A_{ml}\delta_{jk} + A_{jm}A_{mk}\delta_{il}) \end{aligned} \tag{13}$$

In [25], it was proved that the least material symmetry that can be represented by (13) is orthotropy and that the material orthotropy axes coincide with the principal axes of $\mathbf{A}$. In these axes, $\mathbf{A}$ can be written as

$$\mathbf{A} = \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{bmatrix} . \tag{14}$$

There are then 12 parameters to be identified: $\alpha_1, \alpha_2, \alpha_3, a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$. Two cases have to be considered :

- if $\mathbf{A}$ is unknown, Eq. (13) leads to a nonlinear system of nine equations with 12 unknowns. The difficulty encountered to solve this system is a good reason to prefer the formulation (10);
- if $\mathbf{A}$ is known a priori, Eq. (13) then reduces to a regular system. However, the preliminary identification of $\mathbf{A}$ may not be simple : not only the principal directions of $\mathbf{A}$ have to be determined but also the eigenvalues $\alpha_1, \alpha_2, \alpha_3$, i.e. the respective influence of each orthotropy direction. This non trivial identification stage is not necessary in the method proposed in this paper.

Remark 2
Equation (10) allows one to model different levels of initial anisotropy:

- orthotropy, requiring the identification of the nine parameters $\bar{a}_i, \bar{b}_i, \bar{c}_i$ $(i = 1, 2, 3)$;
- tetragonal symmetry (orthotropic symmetry and equivalence between two orthogonal axes, e.g. 1 and 2), when $\bar{a}_1 = \bar{a}_2$, $\bar{b}_1 = \bar{b}_2$, $\bar{c}_1 = \bar{c}_2$;
- cubic symmetry (orthotropic symmetry and equivalence between the three symmetry axes), when $\bar{a}_1 = \bar{a}_2 = \bar{a}_3$, $\bar{b}_1 = \bar{b}_2 = \bar{b}_3$, $\bar{c}_1 = \bar{c}_2 = \bar{c}_3$;
- transverse isotropy (e.g, with respect to axis 1), when $\bar{a}_2 = \bar{a}_3$, $\bar{b}_2 = \bar{b}_3 = 0$, $\bar{c}_2 = \bar{c}_3$,
- isotropy when $\bar{a}_1 = \bar{a}_2 = \bar{a}_3$, $\bar{b}_1 = \bar{b}_2 = \bar{b}_3 = 0$, $\bar{c}_1 = \bar{c}_2 = \bar{c}_3$. The free energy density w^0 then reduces to the classical expression with two parameters, λ and μ Eq. (2).

However, because of the limited number of invariants entering Eq. (10), orthotropy is the weakest material symmetry (the strongest anisotropy) that can be modelled by the simplified method proposed here.

In order to illustrate the identification procedure, consider the example of a bi-directional (0°, 90°) ceramic-ceramic composite produced by SEP. It consists of 2D plates of a chemical vapor-infiltration processed SiC matrix reinforced with plies of Nicalon fibers. The components of the stiffness tensor are identified by an ultrasonic evaluation technique, [27], which makes it possible to measure the nine stiffness coefficients describing orthotropy. If the fiber directions are parallel to axes x_2 and x_3, the C_{ijkl} components are:

$$\begin{array}{lll} C_{1111} = 169\,\text{GPa}, & C_{1122} = 48\,\text{GPa}, & C_{1212} = 57\,\text{GPa}, \\ C_{2222} = 295\,\text{GPa}, & C_{2233} = 145\,\text{GPa}, & C_{2323} = 93\,\text{GPa}, \\ C_{3333} = 318\,\text{GPa}, & C_{1133} = 51\,\text{GPa}, & C_{1313} = 60\,\text{GPa} \end{array},$$

Note that the material is orthotropic (and not transversely isotropic) since

$$C_{2222} \neq C_{3333}, C_{1212} \neq C_{1313}, C_{1122} \neq C_{1133} \text{ and } C_{2323} \neq \frac{1}{2}(C_{2222} - C_{2233}).$$

Solving the system (12) with the values for C_{ijkl} leads to the following set of parameters $(\bar{a}_i, \bar{b}_i, \bar{c}_i)$, $i = 1, 2, 3$.

$$\begin{array}{lll} \bar{a}_1 = -23.0\,\text{GPa}, & \bar{a}_2 = 71.0\,\text{GPa}, & \bar{a}_3 = 74.0\,\text{GPa}, \\ \bar{b}_1 = 83.5\,\text{GPa}, & \bar{b}_2 = -13.5\,\text{GPa}, & \bar{b}_3 = -11.0\,\text{GPa}, \\ \bar{c}_1 = 24.0\,\text{GPa}, & \bar{c}_2 = 90.0\,\text{GPa}, & \bar{c}_3 = 96.0\ \ \text{GPa} \end{array}.$$

In Sec. 3, tension tests on plates made of this material will be simulated.

3
Anisotropic damage effects

The previous section was concerned with the elastic initial anisotropy (primary anisotropy). This anisotropy may strongly affect the matrix-cracking mechanism. The aim of this section is to propose a model accounting for primary anisotropy, mesocrack growth and interaction between both kinds of anisotropy (initial and induced by the presence of defects). The model is assumed to concern material degradation mechanisms not exhibiting notable irreversible stress/strain effects after loading-unloading cycles. This is especially valid for brittle matrix composites when the matrix alone is damaged. A contrary mechanism is e.g. residual strain caused by fiber debonding and sliding at the fiber/matrix interface in fiber-reinforced composites. These mechanisms follow the matrix cracking and intervene at an advanced stage of loading; they are not the subject of this section.

3.1
Damage variable and thermodynamic potential

A single internal variable describes the generation and growth of mesocracks since this mechanism is the only dissipative phenomenon considered. A second-order tensor $\mathbf{D}$ is chosen as damage variable to account for defect orientation

$$\mathbf{D} = \sum_i d^i(S)\mathbf{n}^i \otimes \mathbf{n}^i \quad , \tag{15}$$

where $\mathbf{n}^i$ is the normal to the i-th system of parallel mesocracks and $d^i(S)$ a dimensionless scalar function accounting for the extent of the cracks. Note that $\mathbf{D}$ is a symmetric second–order tensor; it has three positive eigenvalues and three orthogonal eigenvectors. This means that any system of microcracks decomposed into $1, \ldots, n$ subsystems of parallel cracks can be reduced to three equivalent orthogonal sets of cracks. The presence of microcracks thus induces a particular form of orthotropy. This form of the damage variable is motivated by micromechanics, e.g. [28]. However, the term *d(S)* has to be considered here as a macroscopic parameter and not as a microscopic quantity, unlike the classical crack density parameter $\sum_i (l^i)^3/V$ for circular cracks of radii l^i in a representative volume element V.

The basic version of the model at stake [12] assumes the initial isotropy of the material and postulates that any damage configuration is described by the single variable $\mathbf{D}$. Micromechanical studies, [28], assuming noninteraction of penny-shaped cracks in a homogeneous isotropic elastic matrix, show that damage should be rigorously characterized by two damage parameters, namely $\mathbf{D}$ and its extension to the fourth-order, $\sum_i d^i(S)\mathbf{n}^i \otimes \mathbf{n}^i \otimes \mathbf{n}^i \otimes \mathbf{n}^i$. However, when cracks are open (i.e. active), the influence of the term involving the fourth-order tensor is negligible with respect to that of the $\mathbf{D}$-term. The corresponding term should be included when crack closure intervenes, [13], i.e. for example in the case of cyclic loading. For the sake of simplicity, only active damage is considered here and the previous fourth-order term will not enter the constitutive equations. Accounting for damage deactivation by microcrack closure can be done directly by the procedure developed in [13].

While the above conventional damage parameter $\mathbf{D}$ *alone* is sufficient to deal with active damage for initially isotropic materials, some works, [17], prove that it should be accompanied with further insight into cracking when considering anisotropic materials. The enhanced proper crack density parameter that adjusts "relative weight" of a given crack system 'k' according to its *orientation* with respect to the matrix is, for each crack, a fourth-order tensor proportional to $\mathbf{n}^{(k)} \otimes \mathbf{B}^{(k)} \otimes \mathbf{n}^{(k)}$, see [17], where $\mathbf{n}^{(k)}$ is the normal to the k-th crack and $\mathbf{B}^{(k)}$ is the crack-opening displacement second-order tensor related to the system k, namely the tensor which links the average displacement discontinuity vector to the traction vector. The tensor $\mathbf{B}$ reflects the fact that e.g. cracks normal to the stiffer direction of the matrix produce a stronger impact on the effective stiffness than the ones normal to the softer direction. It depends on the crack orientation with respect to the anisotropy axes of the matrix. Its expression is difficult to find in the closed form in the most general case.

An alternative approach advanced here circumvents this difficulty by keeping a macroscopic, while physically motivated, formulation: no reference is made to the exact micromechanical form of $\mathbf{n} \otimes \mathbf{B} \otimes \mathbf{n}$ (which is anyway hardly known) while a macroscopic fourth-order tensor involving both $\mathbf{D}$ (characterizing orientation and extend of the crack array) and $\mathbf{A}$ (i.e. the orthotropy direction, see previous section) enters explicitly the equations of the model. The form of this term is chosen by extension of the basic version of the model dealing with initially isotropic materials. Indeed, the stiffness tensor $\mathbf{C}$ related to this version is:

$$\mathbf{C} = \mathbf{C}^0 + \alpha(\mathbf{I} \otimes \mathbf{D} + \mathbf{D} \otimes \mathbf{I}) + 2\beta(\mathbf{I} \,\overline{\underline{\otimes}}\, \mathbf{D} + \mathbf{D} \,\overline{\underline{\otimes}}\, \mathbf{I}) \quad , \tag{16}$$

where $\mathbf{C}^0$ is the initial elastic stiffness tensor, α and β are material parameters and the tensor products $\otimes$ and $\overline{\underline{\otimes}}$ are defined by:

$$\begin{aligned} (\mathbf{a} \otimes \mathbf{b})_{ijkl} &= a_{ij} b_{kl}, \\ (\mathbf{a} \,\overline{\underline{\otimes}}\, \mathbf{b})_{ijkl} &= \tfrac{1}{2}\left(a_{ik} b_{jl} + a_{il} b_{jk}\right) \quad . \end{aligned} \tag{17}$$

The anisotropic enhanced version accounting through $\mathbf{D}$, for the relative weight of equivalent crack systems with respect to primary anisotropy axes is obtained by replacing in (16) the identity ("isotropic") tensor $\mathbf{I}$ by the orientation ("anisotropic") tensors $\mathbf{A}_i$:

$$\mathbf{C} = \mathbf{C}^0 + \sum_{i=1}^{3} \alpha_i(\mathbf{A}_i \otimes \mathbf{D} + \mathbf{D} \otimes \mathbf{A}_i) + 2\beta_i(\mathbf{A}_i \,\overline{\underline{\otimes}}\, \mathbf{D} + \mathbf{D} \,\overline{\underline{\otimes}}\, \mathbf{A}_i) = \mathbf{C}^0 + \Delta\mathbf{C} \quad . \tag{18}$$

Now, the initial $\mathbf{A}_i$-embodied anisotropy co-exists with the damage-induced one: the fourth-order tensors involving $\mathbf{A}_i$ and $\mathbf{D}$ combine initial orthotropy and evolving damage effects. The expression of $\Delta\mathbf{C}$, like that of $\mathbf{B}$, contains information on damage and primary anisotropy. The group of elastic symmetry of the properties is an intersection of the group of symmetry of $\mathbf{C}^0$ (orthotropy of the matrix without cracks) and the one characterizing $\Delta\mathbf{C}$. If the principal axes of $\mathbf{D}$ coincide with the orthotropy axes of $\mathbf{C}^0$, the material remains orthotropic. If they do not coincide, the effective properties have no elements of symmetry. Six parameters, α_i, β_i, $i = 1, 2, 3$, have to be identified as compared to two in the basic version. Expression (18) leads to the following thermodynamic potential (free energy per unit volume):

$$w(\boldsymbol{\varepsilon}, \mathbf{D}; \mathbf{A}_i) = \sum_{i=1}^{3} \left[\bar{a}_i \mathrm{tr}\,\boldsymbol{\varepsilon}\,\mathrm{tr}(\mathbf{A}_i.\boldsymbol{\varepsilon}) + \bar{b}_i[\mathrm{tr}(\mathbf{A}_i.\boldsymbol{\varepsilon})]^2 + \bar{c}_i \mathrm{tr}(\mathbf{A}_i.\boldsymbol{\varepsilon}.\boldsymbol{\varepsilon})\right] + \sum_{i=1}^{3} [\alpha_i \mathrm{tr}(\boldsymbol{\varepsilon}.\mathbf{A}_i)\mathrm{tr}(\boldsymbol{\varepsilon}.\mathbf{D}) + 2\beta_i \mathrm{tr}(\boldsymbol{\varepsilon}.\mathbf{A}_i.\boldsymbol{\varepsilon}.\mathbf{D})] \ . \quad (19)$$

The second term represents the variation of free energy due to damage and the effects of interaction of primary anisotropy with the damage-induced one, leading eventually to further loss of material symmetry. Unlike the expression of the energy for the basic version of the reference model, [12], the term $g\,\mathrm{tr}(\boldsymbol{\varepsilon}.\mathbf{D})$ giving rise to residual macroscopic stress for $\boldsymbol{\varepsilon} = \mathbf{0}$ (and, dually, $\boldsymbol{\varepsilon} \neq \mathbf{0}$ for $\boldsymbol{\sigma} = \mathbf{0}$) is not considered in Eq. (19).

The corresponding elastic stress $\boldsymbol{\sigma}$ and thermodynamic force $\mathbf{F}^D$ related to damage are determined by partial derivation of w:

$$\boldsymbol{\sigma} = \frac{\partial w}{\partial \boldsymbol{\varepsilon}} = \sum_{i=1}^{3} \left\{\bar{a}_i[\mathrm{tr}(\mathbf{A}_i.\boldsymbol{\varepsilon})\mathbf{I} + (\mathrm{tr}\,\boldsymbol{\varepsilon})\mathbf{A}_i] + 2\bar{b}_i \mathrm{tr}(\mathbf{A}_i.\boldsymbol{\varepsilon})\mathbf{A}_i + \bar{c}_i(\mathbf{A}_i.\boldsymbol{\varepsilon} + \boldsymbol{\varepsilon}.\mathbf{A}_i)\right\} + \sum_{i=1}^{3} \{\alpha_i[\mathrm{tr}(\boldsymbol{\varepsilon}.\mathbf{D})\mathbf{A}_i + \mathrm{tr}(\boldsymbol{\varepsilon}.\mathbf{A}_i)\mathbf{D}] + 2\beta_i(\mathbf{A}_i.\boldsymbol{\varepsilon}.\mathbf{D} + \mathbf{D}.\boldsymbol{\varepsilon}.\mathbf{A}_i)\} \ , \quad (20)$$

$$\mathbf{F}^D = -\frac{\partial w}{\partial \mathbf{D}} = \sum_{i=1}^{3} [-\alpha_i \mathrm{tr}(\boldsymbol{\varepsilon}.\mathbf{A}_i)\boldsymbol{\varepsilon} - 2\beta_i \boldsymbol{\varepsilon}.\mathbf{A}_i.\boldsymbol{\varepsilon}] \ , \quad (21)$$

where $\mathbf{F}^D$ can be interpreted as the damage energy release rate.

Note that there exists an explicit relation between the parameters α_i and β_i, the matrix constants and the damage variable. The parameters α_i and β_i are related to the influence of damage on the elastic moduli: they represent fundamental properties of the material. For example, consider a shear loading ($\sigma_{12} = \sigma_{21} = \tau$, others $\sigma_{ij} = 0$) with the following damage configuration: $D_{11} = D$, others $D_{ij} = 0$ (a single crack system normal to the direction 1). Equation (20) leads to the following expression for τ:

$$\tau = 2G_{12}^0 \varepsilon_{12} + 2\beta_1 \varepsilon_{12} D + 2\beta_2 \varepsilon_{12} D \ , \quad (22)$$

where G_{12}^0 is the shear modulus without damage effect. The global shear modulus G_{12} is then

$$G_{12} = G_{12}^0 + (\beta_1 + \beta_2)D \ . \quad (23)$$

The coefficients β_i directly affect the shear moduli. Moreover, the Young moduli are altered by a combination of α_i and β_i, the respective expressions can be easily exemplified and are not explicitly cited in the paper.

3.2
Damage evolution law

The threshold $f = 0$ delimiting the elastic domain is expressed in the proper space of components of $\mathbf{F}^D$, the thermodynamic force related to $\mathbf{D}$ (damage driving force). The evolution of $\mathbf{D}$ is assumed to follow the normality rule, corresponding to the principle of maximum dissipation and exhibiting splitting-like damage mechanism commonly observed in brittle solids. However, unlike the classical standard models, f is a function of a part of $\mathbf{F}^D$ only: experiments, [29], give evidence to the role played by the positive strain during the damage process. That is why the damage criterion $f(\mathbf{F}^D, \mathbf{D}) = 0$ is chosen explicitly dependent on the positive part $\mathbf{F}^{D+}$ of $\mathbf{F}^D$:

$$\mathbf{F}^{D+} = \sum_{i=1}^{3}[-\alpha_i \mathrm{tr}(\boldsymbol{\varepsilon}.\mathrm{A}_i)\boldsymbol{\varepsilon}^+ - 2\beta_i\boldsymbol{\varepsilon}^+.\mathbf{A}_i.\boldsymbol{\varepsilon}^+]; \quad \mathbf{F}^{D-} = \mathbf{F}^D - \mathbf{F}^{D+}; \quad \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^+ + \boldsymbol{\varepsilon}^- \ , \tag{24}$$

where $\mathbf{F}^{D+}$ is defined as a part of $\mathbf{F}^D$ involving $\boldsymbol{\varepsilon}^+$ (positive strain), which is built by extracting the positive eigenvalues of $\boldsymbol{\varepsilon}$ (see [30, 13] for explicit construction of $\boldsymbol{\varepsilon}^+$). One assumes that the part $\mathbf{F}^{D+}$ plays a key role in the damage criterion:

$$f(\mathbf{F}^D - \mathbf{F}^{D-}, \mathbf{D}) = \sqrt{\frac{1}{2}\mathrm{tr}[(\mathbf{F}^D - \mathbf{F}^{D-})\cdot(\mathbf{F}^D - \mathbf{F}^{D-})]} + B\mathrm{tr}\left[(\mathbf{F}^D - \mathbf{F}^{D-})\cdot\mathbf{D}\right] - (C_0 + C_1 \mathrm{tr}\mathbf{D}) = 0 \ , \tag{25}$$

where C_0 is the initial damage threshold, while C_1 and B are related to the evolution of the surface $f = 0$ when $\mathbf{D}$ evolves. The rate-independent damage evolution law is written as follows:

$$\dot{\mathbf{D}} = \begin{cases} 0 & \text{if } f < 0 \text{ or } f = 0, \dot{f} < 0, \\ \Lambda_D \frac{\partial f}{\partial \mathbf{F}^D} = \Lambda_D\left[\frac{\mathbf{F}^{D+}}{\sqrt{2\mathrm{tr}(\mathbf{F}^{D+}.\mathbf{F}^{D+})}} + B\mathbf{D}\right] & \text{if } f = 0 \text{ and } \dot{f} = 0, \Lambda_D \geq 0 \ . \end{cases} \tag{26}$$

The importance of the positive strain appears in the first term of the evolution law involving $\mathbf{F}^{D+}$. The second term $B\mathbf{D}$ is called the drag-term, it represents the influence of the current value of damage on its instantaneous evolution, see also [15].

3.3 Dissipation

The choice of the decomposition (24) for $\mathbf{F}^D$ has been made on the basis of thermodynamic considerations. Let ϕ be the intrinsic dissipation due to mesocrack growth:

$$\phi = \mathbf{F}^D : \dot{\mathbf{D}} \ . \tag{27}$$

Let us consider for simplicity the case $B = 0$. Equations (21) and (26) lead to the following expression of dissipation:

$$\phi = \Lambda_D \frac{\mathrm{tr}(\mathbf{F}^D.\mathbf{F}^{D+})}{\sqrt{2\mathrm{tr}(\mathbf{F}^{D+}.\mathbf{F}^{D+})}} \ . \tag{28}$$

Since the decomposition (24) gives rise to the orthogonality of $\mathbf{F}^{D+}$ and $\mathbf{F}^{D-}$ and Λ_D is positive, ϕ has the same sign as $\mathbf{F}^{D+} : \mathbf{F}^{D+}$, i.e. ϕ is always positive: the second law of thermodynamics is thus satisfied.

When B is different from zero, the sign of ϕ becomes undetermined for any expression of the part of $\mathbf{F}^D$ entering Eq. (25). However, the expression (25) of the damage criterion ensures the convexity of the loading surface (and then the non-negative values of ϕ), provided the absolute value of B is less than a given limit depending on the damage level. A series of simulations has allowed to find a bound to the value of damage and to conclude that the dissipation always remains positive when $|B|$ does not exceed $\sqrt{2}/2$. Moreover, B has to be negative, since $B\mathbf{D}$ in (24) is a drag-term. In conclusion, the decomposition (24) and the expression (25) allow to ensure non-negative values of ϕ provided $-\frac{\sqrt{2}}{2} \leq B \leq 0$.

3.4 Example

The predictive capacity of the presented approach is tested by simulating tension tests on the composite whose elastic constants have been determined in Sec. 2.2. The experimental data are those of [16] and [31]. They consist of tension tests on bidirectional $(0°, 90°)$ woven ceramic matrix plates, with different orientations θ of the tension axis with respect to fiber axes, see Fig. 1.

The above cited references provide tension tests for $\theta = 0°$, $20°$ and $45°$. Parameters $C_0, C_1, B, \alpha_2, \alpha_3, \beta_2, \beta_3$ are identified on the 0°- and 45°-tests. The third tension test (20°) is used as a preliminary validation test. Note that no experimental information on the direction 1 is available. Thus, in this particular case, the parameters α_1 and β_1 are arbitrarily chosen to be equal to zero. Table 1 collects the values of the parameters.

Figures 2–4 exhibit a fair correlation between the experimental data and the corresponding simulation (σ_{33} in MPa). Note, in particular, the respective position (Fig. 5) of each curve in agreement with the experiment, [16]: even if the elastic response of the composite for $\theta = 45°$ is initially stiffer than the response in the direction parallel to fibers ($\theta = 0°$), the stress level is lower for $\theta = 45°$ than for $\theta = 0°$ or 20°. This illustrates the interaction effect between (oriented) cracking and the primary orthotropy on resultant material degradation in accordance with the postulates formulated in Sec. 3.1.

4
Conclusion and prospective work

The purpose of this work was to complete a three-dimensional damage modelling framework developed in earlier works [12–15] for a large class of quasi-brittle solids. It aimed at

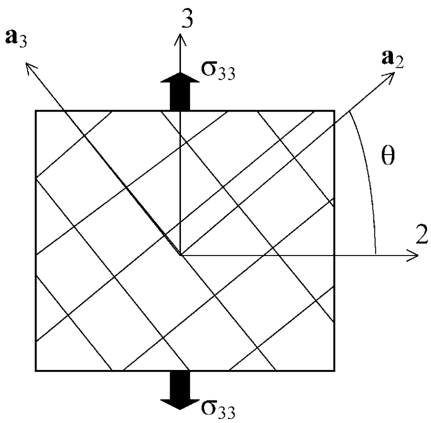

Fig. 1. Schematization of the plate under tension

Table 1. Material constants for a class of CMC

C_0 (MPa)	C_1 (MPa)	B (1)	α_2 (MPa)	β_2 (MPa)	α_3 (MPa)	β_3 (MPa)
0.017	0.058	−0.7	0.	−62000.	0.	−48000.

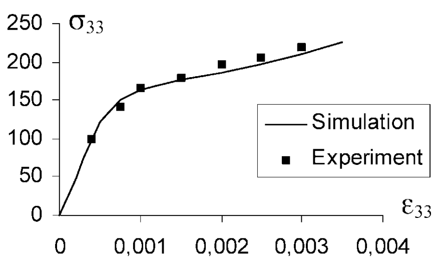

Fig. 2. Comparison simulation/experiment $\theta = 0°$

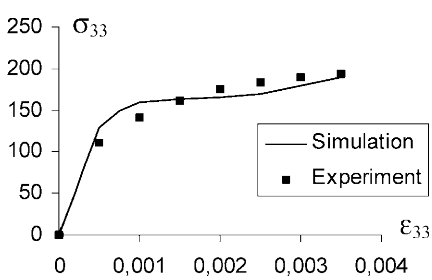

Fig. 3. Comparison simulation/experiment $\theta = 45°$

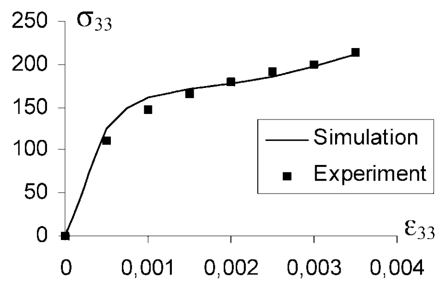

Fig. 4. Comparison simulation/experiment $\theta = 20^{\circ}$

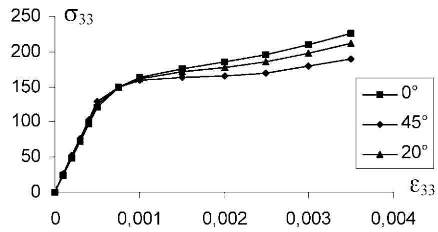

Fig. 5. Respective position of the simulated response for $\theta = 0^{\circ}$, 20° and 45°

adjoining interactional effects between damage-induced orthotropy (a salient effect of the existing model) and an initial anisotropy introduced in this paper. This has been done by pursuing a method which can be considered as phenomenological yet micromechanically motivated, in the spirit of the unified approach presented in the survey article [34]. The existing framework includes damage growth by oriented microcracking, effects of opening/closure (and inverse) transition for microcrack sets and complementary dissipation effects due to frictional resistance and sliding on closed crack sets. These problems are approached within the rate-type constitutive theory with internal variables. The extension proposed in the present paper concerns quasi-brittle solids exhibiting a marked initial anisotropy independently of the secondary, damage-induced one. The initial orthotropy has been introduced here in a particular thrifty manner, allowing for tractable identification due to a reduced number of material constants as compared to other existing schemes. The model allows, furthermore, to take into account major coupling effects between the primary anisotropy and the secondary, damage-induced and evolving anisotropy. This has been done by introducing invariant terms involving simultaneously damage tensor and material fabric tensor in the representation of the free energy as a thermodynamic potential, Eq. (19). This representation is an alternative to the micromechanical expression postulated in [17] comprising the crack-compliance tensor **B** and indicators of crack orientation. An illustration of the interactional effects at stake (primary vs. damage-induced anisotropy) has been given for a brittle matrix fiber-reinforced composite.

The actual completion of the damage modelling framework [12–15] maintains a modular structure of the framework, adding an ingredient to existing modelling segments. They are built to provide tractable tools for efficient structural analysis, see [15, 34], regarding numerical integration and implementation criteria.

Prospective work concerns some adaptation of the above-mentioned complementary dissipative model, [14], involving frictional sliding on closed cracks to brittle composites with structural (primary) anisotropy focused on by the present paper. In particular, for fiber reinforced composites, sliding at fiber/matrix interface as well as (final) fiber breakage are known as successive damage stages leading to failure, see, e.g. [32, 33]. The modelling framework presented can offer an alternative to existing approaches, [4, 35] due to vigorous search of an optimum regarding tractability (a number of constants to be identified) vs. pertinence considerations. For example, to deal with plasticity-like slip on fiber/matrix interfaces, one can further implement the modelling segment presented in this paper. Unlike for the microcrack sliding, the fiber/matrix sliding occurs in fixed (known) directions $\mathbf{a}_i$ $(i = 1, 2, 3)$ interpreted in Sec. 2 with symmetric fabric (orientation) tensors $\mathbf{A}_i$. A scalar internal variable γ_i $(i = 1, 2, 3)$, representing relative interfacial sliding, associated with respective tensor $\mathbf{A}_i$ is combined with the invariant $\mathrm{tr}(\boldsymbol{\varepsilon}.\mathbf{A}_i)$ to generate residual effects due to frictional slip in the way shown in [14]. The form of the energy w given above by Eq. (19) is then completed to yield the following:

$$w(\boldsymbol{\varepsilon},\mathbf{D},\gamma_i;\mathbf{A}_i)=\sum_{i=1}^{3}\left[\bar{a}_i\mathrm{tr}\,\boldsymbol{\varepsilon}\,\mathrm{tr}(\mathbf{A}_i.\boldsymbol{\varepsilon})+\bar{b}_i[\mathrm{tr}(\mathbf{A}_i.\boldsymbol{\varepsilon})]^2+\bar{c}_i\mathrm{tr}(\mathbf{A}_i.\boldsymbol{\varepsilon}.\boldsymbol{\varepsilon})\right]$$

$$+\sum_{i=1}^{3}[\alpha_i\mathrm{tr}(\boldsymbol{\varepsilon}.\mathbf{A}_i)\mathrm{tr}(\boldsymbol{\varepsilon}.\mathbf{D})+2\beta_i\mathrm{tr}(\boldsymbol{\varepsilon}.\mathbf{A}_i.\boldsymbol{\varepsilon}.\mathbf{D})]+\sum_{i=1}^{3}\left[\eta_1^i\gamma_i\mathrm{tr}(\boldsymbol{\varepsilon}.\mathbf{A}_i)+\eta_2^i\gamma_i^2\right], \quad (29)$$

where η_1^i and η_2^i $(i=1,2,3)$ are parameters to be identified. From Eq. (29) one can obtain the novel expression for stress $\boldsymbol{\sigma}$:

$$\boldsymbol{\sigma}=\frac{\partial w}{\partial\boldsymbol{\varepsilon}}=\sum_{i=1}^{3}\left[\bar{a}_i[\mathrm{tr}(\mathbf{A}_i.\boldsymbol{\varepsilon})\mathbf{I}+(\mathrm{tr}\,\boldsymbol{\varepsilon})\mathbf{A}_i]+2\bar{b}_i\mathrm{tr}(\mathbf{A}_i.\boldsymbol{\varepsilon})\mathbf{A}_i+\bar{c}_i(\boldsymbol{\varepsilon}.\mathbf{A}_i+\mathbf{A}_i.\boldsymbol{\varepsilon})\right]$$

$$+\sum_{i=1}^{3}\{\alpha_i[\mathrm{tr}(\boldsymbol{\varepsilon}.\mathbf{D})\mathbf{A}_i+\mathrm{tr}(\boldsymbol{\varepsilon}.\mathbf{A}_i)\mathbf{D}]+2\beta_i(\mathbf{A}_i.\boldsymbol{\varepsilon}.\mathbf{D}+\mathbf{D}.\boldsymbol{\varepsilon}.\mathbf{A}_i)\}+\sum_{i=1}^{3}\eta_1^i\gamma_i\mathbf{A}_i \quad (30)$$

Beside the classical terms similar to those of (19), the third term of (29) includes energy terms relative to sliding-induced plasticity effects due to friction at the interface fiber/matrix when the fiber is debonding in slip-like manner. Further work will attempt to propose an adequate law for the evolution of the variables γ_i.

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